NOTE ON THE INVARIANT TOTAL DIFFERENTIAL EQUATION $Pdx + Qdy + Rdz = 0$.

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If we write, as is usual,

$$\frac{dy}{dx} = y_1, \quad \frac{dz}{dx} \equiv z_1,$$

Lie has shown that all equations of the general form

$$\Omega(x, y, z, y_1, z_1) = 0,$$

which are invariant under a given infinitesimal transformation

$$Uf \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial z},$$

are found as follows. We form the expressions

$$\eta_1 \equiv \frac{d\eta}{dx} - y_1 \frac{d\xi}{dx}, \quad \zeta_1 \equiv \frac{d\zeta}{dx} - y_1 \frac{d\xi}{dx};$$

and find the solutions of the linear partial differential equation of the first order in five variables,

$$U_1 f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta \frac{\partial f}{\partial z} + \eta_1 \frac{\partial f}{\partial y_1} + \zeta_1 \frac{\partial f}{\partial z_1} = 0.$$

Let these solutions be, respectively,

$$u(x, y, z), \quad v(x, y, z), \quad u_1(x, y, z, y_1, z_1), \quad v_1(x, y, z, y_1, z_1);$$

then the most general equation in the variables x, y, z, y_1, z_1, which is invariant under Uf has the form

$$F(u, v, u_1, v_1) = 0.$$

Now if the functions F, u_1, and v_1 are so chosen that the equation $F = 0$ is *linear* in terms of y_1 and z_1, it is clear that we thus obtain a *total* differential equation in the variables x, y, z,—which is invariant under Uf.

As a simple example, it may be at once verified that the most general equation $F = 0$ which is invariant under the *perspective* transformation

$$Uf \equiv x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} + z \frac{\partial f}{\partial z},$$

has the form

$$F\left[\frac{y}{x}, \frac{z}{x}, y_1, z_1\right] = 0 .$$

Hence, all total differential equations of the form

$$P\left[\frac{y}{x}, \frac{z}{x}\right] dx + Q\left[\frac{y}{x}, \frac{z}{x}\right] dy + R\left[\frac{y}{x}, \frac{z}{x}\right] dz = 0 ,$$

that is, all *homogeneous* total differential equations are invariant under the perspective transformation.

In an analogous manner, we might proceed to establish an unlimited number of *types* of total differential equations, which are invariant under given infinitesimal transformations.

We shall now suppose that a total differential equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

is given, and shall assume that this equation is invariant under a known infinitesimal transformation Uf. Two essentially different cases present themselves, according to whether the equation (1) satisfies the condition of integrability or not. We shall take up the latter case first.

I: If the invariant total differential equation (1) does not satisfy the condition of integrability, the *general* solution of this equation can always be obtained by a quadrature.

For the general solution of any non-integrable equation (1) is obtained by associating with (1) an equation of the form $\psi(x, y, z) = 0$, where ψ is an arbitrary function. But since u and v, the solutions of $Uf = 0$, are independent functions, it is readily seen that the general solution of (1) can always be obtained by associating with (1) an equation of the form

$$\Phi(u, v) = 0 ,$$

where Φ is an arbitrary function of u and v. Hence we obtain a simultaneous system consisting of the equations (1) and

$$d\Phi - P'dx + Q'dy + R'dz = 0 ; \tag{2}$$

and this simultaneous system is, in turn, equivalent to the linear partial differential equation of the first order

$$Af - (QR' - Q'R)\frac{\partial f}{\partial x} + (RP' - R'P)\frac{\partial f}{\partial y} + (PQ' - P'Q)\frac{\partial f}{\partial z} = 0 .$$

Of course $Af = 0$ is invariant, since (1) and (2) are invariant; that is, the

equations $Af = 0$, $Uf = 0$ form a complete system with the known solution $\Phi(u, v)$.

Since the function Φ must contain two, at least, of the variables x, y, z,—we can introduce as new variables, Φ and two of the old variables,—say, x, y, Φ. Also, since $A(\Phi) \equiv U(\Phi) \equiv 0$, it is clear that the equation $Af = 0$ becomes, in the new variables, a linear partial differential equation of the first order in x and y, which is invariant under a known transformation Uf in the same variables.* Hence, another solution of $Af = 0$ which is independent of Φ,—that is, the general solution of (1) with the condition $\Phi = 0$, is obtained from an ordinary differential equation of the first order in two variables by a quadrature.

II: If the given invariant total differential equation

$$Pdx + Qdy + Rdz = 0 \tag{1}$$

satisfies the condition of integrability;—that is, if the general solution of (1) can be expressed by means of a single equation of the form

$$\Omega(x, y, z) = c,$$

then it is clear that Ω, the integral-function of (1), will be at the same time the common solution of the equations

$$P\frac{\partial f}{\partial y} - Q\frac{\partial f}{\partial x} = 0, \quad P\frac{\partial f}{\partial z} - R\frac{\partial f}{\partial x} = 0,$$

which, in this case, form a complete system.

Since the family of surfaces $\Omega = c$ is invariant under Uf, the expression $U(\Omega)$ must be a function of Ω alone. If $U(\Omega) \equiv 0$, the surfaces $\Omega = c$ are not transformed by Uf at all; and, in this case, Uf is said to be *trivial* with regard to equation (1). Hence, if we assume, as we always shall do, that Uf is not *trivial*, the expression $U(\Omega)$ is not zero; and we can always choose the function Ω in such manner that $U(\Omega) \equiv 1$.† Thus Ω, the integral-function of (1), satisfies the three equations

$$Q\frac{\partial \Omega}{\partial x} - P\frac{\partial \Omega}{\partial y} = 0,$$

$$R\frac{\partial \Omega}{\partial x} - P\frac{\partial \Omega}{\partial z} \equiv 0,$$

$$\xi\frac{\partial \Omega}{\partial x} + \eta\frac{\partial \Omega}{\partial y} + \zeta\frac{\partial \Omega}{\partial z} = 1.$$

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* Lie's "Differentialgleichungen," p. 437.

† Cf. Lie.

Hence, it is immediately seen that Ω is obtained by a quadrature, in the form,

$$\Omega \equiv \int \frac{Pdx + Qdy + Rdz}{P\xi + Q\eta + R\zeta}.$$

That is, if $P\xi + Q\eta + R\zeta \neq 0$, *the expression*

$$M \equiv \frac{1}{P\xi + Q\eta + R\zeta}$$

is an integrating factor of the invariant integrable equation (1).

If $P\xi + Q\eta + R\zeta \equiv 0$, it is clear that the path-curves of the transformation Uf all lie *on* the surfaces $\Omega = c$; so that Uf is *trivial* with regard to (1),—a case which was excluded.

We may remark that if the integral surfaces of the integrable invariant equation (1) are given in the form

$$\Phi(x, y, z, c) = 0, \qquad c = \text{const.}$$

an envelope of these ∞^1 surfaces may exist. This envelope must be a surface (possibly degenerated), which is clearly invariant under the transformation Uf. That is, the surface must be generated by path-curves of the transformation Uf, defined by the equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}. \tag{3}$$

Thus, since the equation to the envelope satisfies both the equations (3) and the equation (1), the envelope, (or the singular solution of (1)), is given by

$$\omega(x, y, z) \equiv P\xi + Q\eta + R\zeta = 0. \tag{4}$$

On the envelope the integrating factor M is, of course, infinite.

We may further notice that the directions of the characteristics of the envelope are given by the equation (1) in connection with the equation resulting from the total differentiation of the equation $\omega =$ const. That is, the characteristics of the envelope satisfy the equations

$$\begin{aligned} Pdx + Qdy + Rdz &= 0, \\ \omega_x dx + \omega_y dy + \omega_z dz &= 0; \end{aligned} \tag{5}$$

or the equivalent equations

$$\frac{dx}{Q\omega_z - R\omega_y} = \frac{dy}{R\omega_x - P\omega_z} = \frac{dz}{P\omega_y - Q\omega_x}. \tag{6}$$

If a cuspidal edge of the envelope $\omega = 0$ exists, it must clearly be a path-curve of the transformation Uf; that is, from (3) and (6), the cuspidal edge, if one exists, is given by the equations

$$\frac{\xi}{Q\omega_z - R\omega_y} = \frac{\eta}{R\omega_x - P\omega_z} = \frac{\zeta}{P\omega_y - Q\omega_x}.$$

It is easily seen that ξ, η, ζ cannot all vanish either at all points on the envelope, or at all points on the cuspidal edge (if these loci are not degenerated). Hence the envelope of the family of surfaces defined by the integrable total equation (1), which is invariant under a known infinitesimal transformation Uf, can be found without even a quadrature; and the same is true of the cuspidal edge of the envelope. The surfaces themselves can be found by a quadrature.